# Visualization of Complex Integration with GeoGebra

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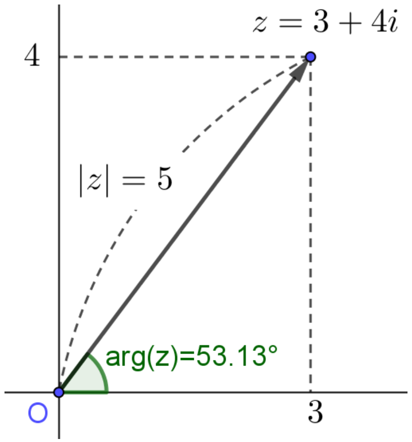
ATCM 2019 in 楽山

Document -> http://mixedmoss.com/temp/complex.pdf
GeoGebra -> http://mixedmoss.com/temp/complex.zip

# § 1. Basic Calculus in the complex plane




Definition of the Complex plane

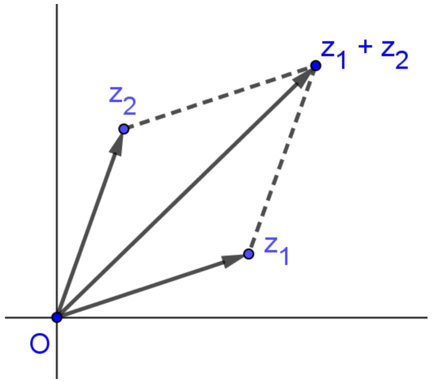
$z = x + iy \ (x, y \in R)$

$\Updownarrow$

$(x, y)$, same as a point in $R^2$

$|z| \cdots$ distance between O and $z$

$\arg(z) \cdots$ angle of the radius O$z$




Addition&Subtraction

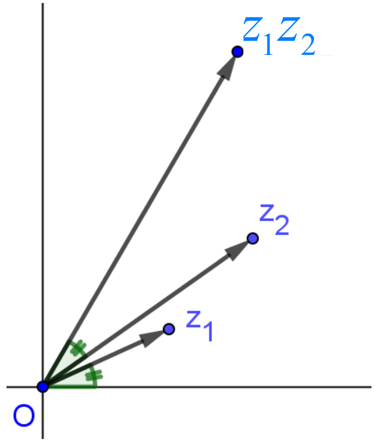
When $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2,$

$$z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2)i$$

$$\Updownarrow$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_1 \\ y_1 + y_2 \end{pmatrix}$$

Same as the sum of 2 vectors.




Multiplication&Division

$$\begin{cases} |z_1 z_2| = |z_1||z_2| \\ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \end{cases}$$

'To multiply $z_1$ to $z_2$' is

'rotate $z_2$ around O by $\arg(z_1)$ &

magnify the distance from O by $|z_1|$ times.'

Example. $(a + bi) \times i = -b + ai$

'To multiply $i$' is 'to rotate by $90^\circ$'

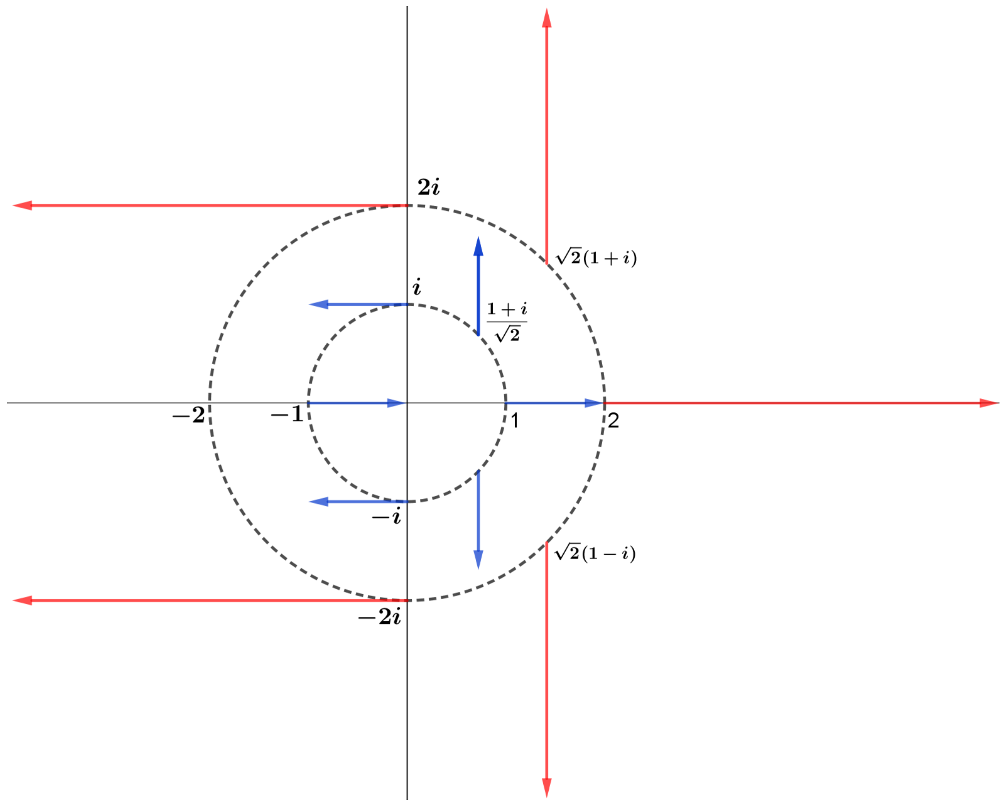
Basic.ggb

# § 2. Complex function as a vector field

A complex function $f(z) = u(z) + v(z)i \quad (z \in C, \quad u(z),\ v(z) \ \in R)$
corresponds to a vector field by assigning a vector $\begin{pmatrix} u(z) \\ v(z) \end{pmatrix}$ to each point $z$.

Example. Vector field of $f(z) = z^2$.

$$\begin{cases} f(\pm 1) = 1,\ \ f(\pm i) = i^2 = -1, \\ f\left(\dfrac{1 \pm i}{\sqrt{2}}\right) = \left(\dfrac{1 \pm i}{\sqrt{2}}\right)^2 = \pm i \\ f(\pm 2) = 4,\ \ f(\pm 2i) = 4i^2 = -4, \\ f\left(\sqrt{2}(1 \pm i)\right) = \pm 4i \\ \quad \vdots \qquad\qquad \vdots \end{cases}$$




Vector field.ggb

# § 3. Definition of complex integration

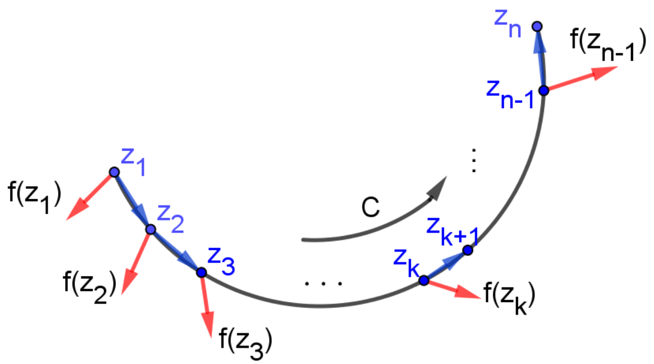
$C$ is a piecewise smooth oriented curve in the complex plane. $z_1, z_2, z_3, \cdots, z_n$ are points on $C$. $\delta$ is the maximum of $|z_{k+1} - z_k|$ $(k = 1, 2, 3 \cdots n)$.

When $\delta$ converges to 0 as $n$ increases, then the line integral of $f(z)$ over $C$ is

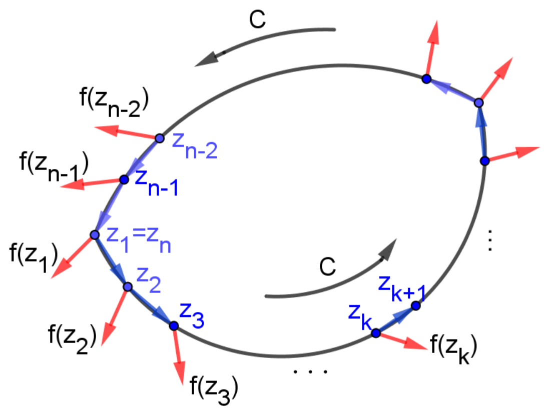
$$\boxed{\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)(z_{k+1} - z_k) = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta z_k}$$

The initial point $z_1$ and the terminal point $z_n$ can coincide. Then $C$ is called a closed curve, and the integral is called a contour integral.

Contour integral is often expressed as $\oint_C f(z)dz$. template@auto.ggb




Line Integral

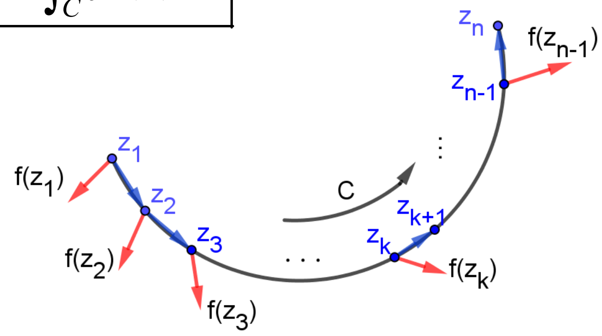


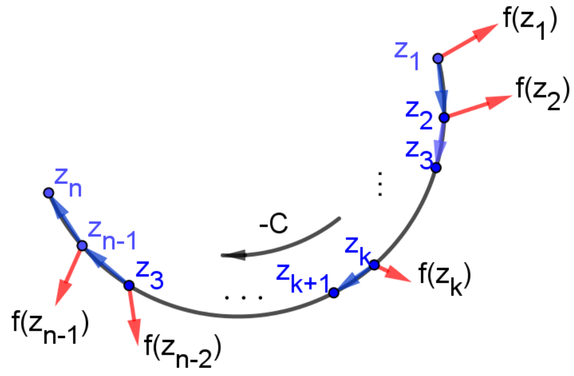

Contour Integral

# Basic properties of complex integration

(1) Orientation reversal. $(-C)$ means the reversed curve of $C$.

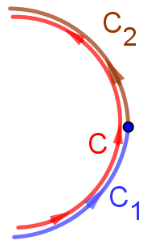
$$\boxed{\int_{-C} f(z)dz = -\int_{C} f(z)dz}$$



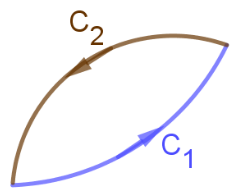




(2) Adding (or partitioning) of path

$$\boxed{\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{C_1+C_2} f(z)dz}$$




$C_1 + C_2 = C$ (semicircle)




$C_1 + C_2$ is closed.

They correspond to these properties in integral calculus.

$$\int_{\beta}^{\alpha} f(x)dx = -\int_{\alpha}^{\beta} f(x)dx,$$

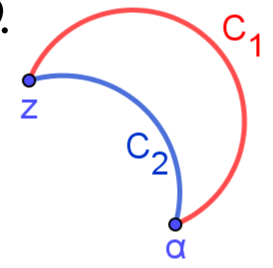
$$\int_{\alpha}^{\beta} f(x)dx + \int_{\beta}^{\gamma} f(x)dx = \int_{\alpha}^{\gamma} f(x)dx$$

# § 4. Fundamental theory of complex integration

$D$ is a domain in the complex plane. $f(z)$ and $F(z)$ are complex functions in $D$.

If $F'(z) = f(z)$ everywhere in $D$, for any curve $C$ $(\in D)$ which joins $\alpha$ & z,

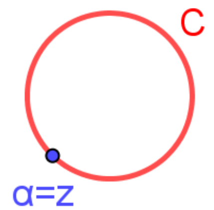
$$\int_C f(z)dz = \left[F(z)\right]_\alpha^z = F(z) - F(\alpha).$$




The value of integral only depends on $\alpha$ & $z$. i.e. It is independent from $C$.

Therefore if $\alpha = z$ (i.e. if $C$ is closed),

$$\oint_c f(z)dz = 0$$

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1overz.;basic.ggb

Example.

Because of $(z^n)' = nz^{n-1} (n = \pm 1, \pm 2, \pm 3, \cdots)$,

$$\cdots, \int_\alpha^z \frac{1}{z^3}dz = -\frac{1}{2z^2} + \frac{1}{2\alpha^2}, \int_\alpha^z \frac{1}{z^2}dz = -\frac{1}{z} + \frac{1}{\alpha}, \int_\alpha^z 1dz = z - \alpha, \int_\alpha^z zdz = \frac{z^2}{2} - \frac{\alpha^2}{2}, \int_\alpha^z z^2dz = \frac{z^3}{3} - \frac{\alpha^3}{3}, \cdots$$

These integrals are independent from the paths.

Note: There is no primitive for $\frac{1}{z}$. Because $\int_\alpha^z \frac{1}{z}dz$ can change with a path $C$ from $\alpha \to z$.

# § 5. Integration of 1/(*z-a*) around a circle

When $z$ is on the cirlcle $C$ of radius $r$, centered at a, anticlockwised, then $z$ is parameterized by

$$z - a = r(\cos\theta + i\sin\theta)$$

$$\therefore dz = r(-\sin\theta + i\cos\theta)d\theta = i(z-a)d\theta$$

1overz@circle.ggb

$$\therefore \frac{dz}{z-a} = id\theta \quad \cdots(!)$$

$|dz| = r\,d\theta = |z-a|\,d\theta$

$(z-a) \xrightarrow{90^\circ \text{ rotate}} dz$

It doesn't depend on $r$.

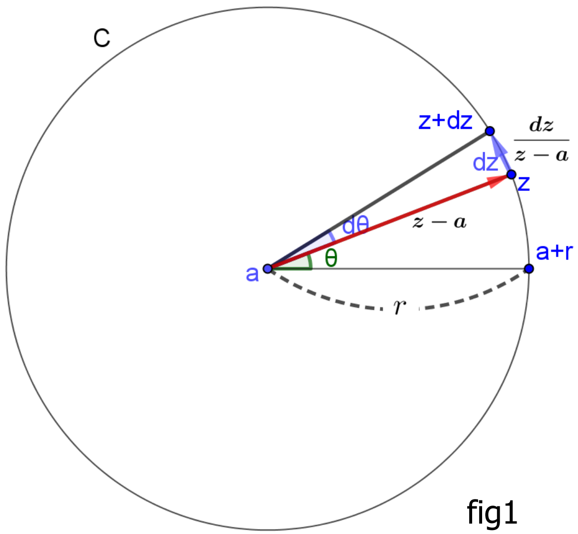


fig1

Therefore, line integral along $C$ (fig2) is,

$$\boxed{\int_C \frac{1}{z-a}dz = \int_0^\theta id\theta = i\theta}$$

Especially when $C$ is a single circle,

$$\boxed{\oint_C \frac{1}{z-a}dz = \int_0^{2\pi} id\theta = 2\pi i}$$

But $F(z) = \int_{a+r}^{z} \frac{1}{z-a}dz$ isn't uniquely defined by $z$.

Line integral along $C'$ (fig3) is

$$\int_{C'} \frac{1}{z-a}dz = \int_0^{\theta+2\pi} id\theta = i\theta + 2\pi i$$

It can be $i(\theta + 2\pi n)$ $(n = 0, \pm1, \pm2 \cdots)$ for the same $z$.

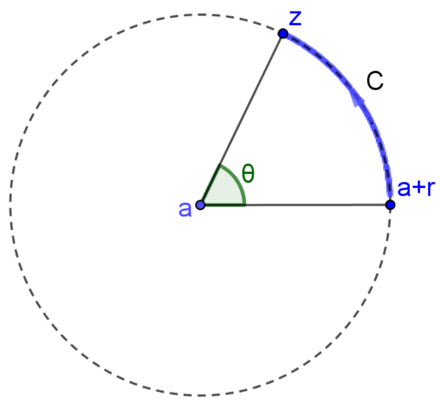


fig2

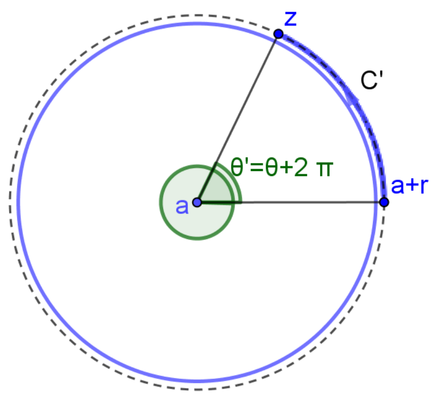


fig3

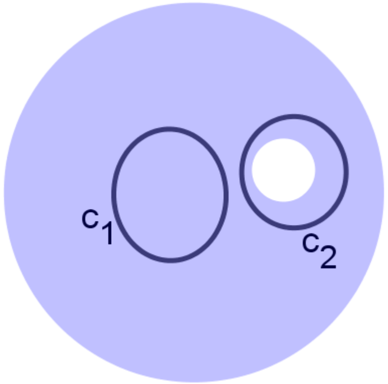
1overz;2circles.ggb

# § 6. *Cauchy* 's Integral theorem

If $f(z)$ is analytic everywhere inside a curve $C$ (i.e. if there is no singular point inside $C$)

$$\oint_c f(z)\,dz = 0$$

- A singular point is a point where $f(z)$ is not analytic.




$f(z)$ is analytic in the shaded area.

*Caushy*'s theorem can be applied to $C_1$.

But it can't be applied to $C_2$. For example,

$\oint_{|z|=1} \frac{1}{z^2}\,dz = 0$ is not implied by *Caushy*'s theorem.

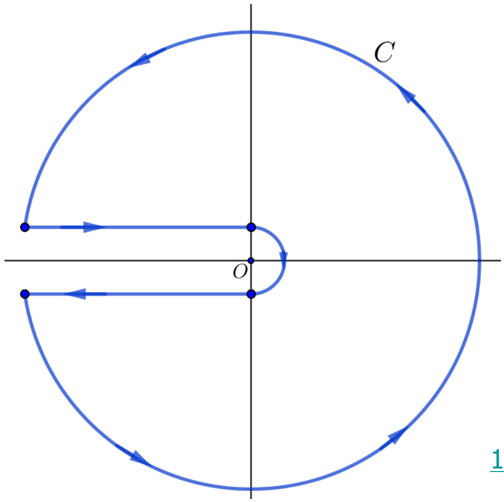
It is implied by the existance of a primitive.

Example.

$f(z) = \dfrac{1}{z}$ is analytic except $O$.

Therfore for any closed curve $C$ which doesn't enclose $O$,

$$\oint_C \frac{1}{z}\,dz = 0$$




1overz;cut plane@auto.ggb

# § 7. Deformation of the path
(Changing end points.)

$f(z)$ is analytic inside a closed curve $C$ except $z_1, z_2$. Devide $C$ into $C_a + C_b$ and make 2 small circles $C_1, C_2$ around $z_1$, $z_2$ all anticlockwise, 4 pathes $T_k$ as below. Then applying *Caushy*'s theorem,

$$\oint_{C_a+T_1-C_1+T_2+C_b+T_3-C_2+T_4} f(z)dz = 0$$

Because line integral is additve,

$$\int_{C_a} f(z)dz + \int_{T_1} f(z)dz - \oint_{C_1} f(z)dz + \int_{T_2} f(z)dz + \int_{C_b} f(z)dz + \int_{T_3} f(z)dz - \oint_{C_2} f(z)dz + \int_{T_4} f(z)dz = 0$$

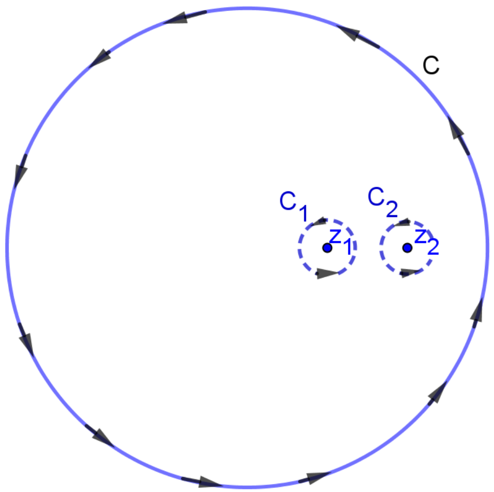
Because $f(z)$ is continuous,

$$\int_{T_1} f(z)dz + \int_{T_2} f(z)dz = 0, \quad \int_{T_3} f(z)dz + \int_{T_4} f(z)dz = 0 \ , \quad \int_{C_a} f(z)dz + \int_{C_b} f(z)dz = \oint_C f(z)dz$$

Therefore,

$$\oint_C f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

$$\therefore \boxed{\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz}$$



# § 8. *Cauchy*'s integral formula

Assume $f(z)$ is analytic inside a closed curve C, points a,z are inside C. Take a small circle $C_a$ of radius $r$ around $a$, anticlockwise. Then by a path deformation,

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_a} \frac{f(z)}{z-a}dz \quad \cdots(1)$$

Because $f(z)$ is analytic, when $r$ is small

$$f(z) \approx f(a) + (z-a)f'(a) \quad (\text{on } C_a)$$

Therefore

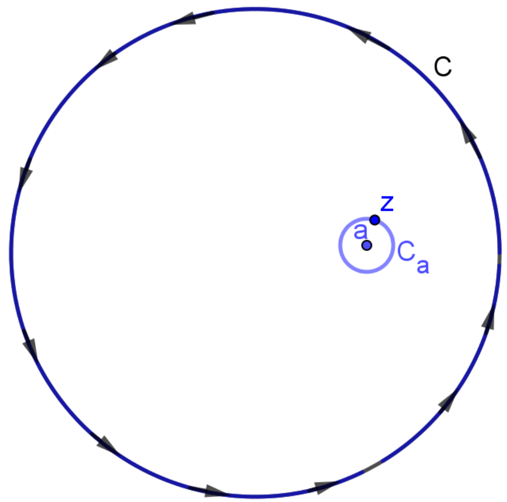
$$\oint_{C_a} \frac{f(z)}{z-a}dz \approx \oint_{C_a} \frac{f(a)+(z-a)f'(a)}{z-a}dz = f(a)\oint_{C_a}\frac{1}{z-a}dz + f'(a)\oint_{C_a}dz = 2\pi i f(a) + 0\cdot f'(a)$$

Because for any $r$, (1) holds true,

$$\oint_C \frac{f(z)}{z-a}dz = \lim_{r\to 0}\oint_{C_a}\frac{f(z)}{z-a}dz = 2\pi i f(a)$$

That is,

$$\boxed{\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)} \quad \left( f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \right)$$




What will happen when $r \to +0$

- "*Cauchy*'s integral formula" is normally written as the upper right form.
- It's easy to extend this to $\dfrac{f(z)}{(z-a)(z-b)}, \cdots, \dfrac{f(z)}{(z-z_1)(z-z_2)\cdots(z-z_n)}$

**Example1** $I = \oint_C \frac{z^2}{(z^2+1)(z^2+4)} dz = \frac{\pi}{3}$ $\left( \begin{array}{l} C\text{:semicircle } \overset{\frown}{AB} \text{ of radius } r > 2 \\ +\text{segment } \overline{BA}, \text{ anticlockwise} \end{array} \right)$

Integrand $f(z) = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}$ has 2 singular points $i$ & $2i$ inside $C$.

Make small circles $C_1, C_2$ around $i$, $2i$, anticlockwise. Then by a path deformation,

$$\oint_C \frac{z^2}{(z^2+1)(z^2+4)} dz = \oint_{C_1} \frac{z^2}{(z^2+1)(z^2+4)} dz + \oint_{C_2} \frac{z^2}{(z^2+1)(z^2+4)} dz$$

Here $f(z)$ can be written in two ways.

$$\frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2/\{(z+i)(z^2+4)\}}{z-i} = \frac{z^2/\{(z^2+1)(z^2+2i)\}}{z-2i}$$

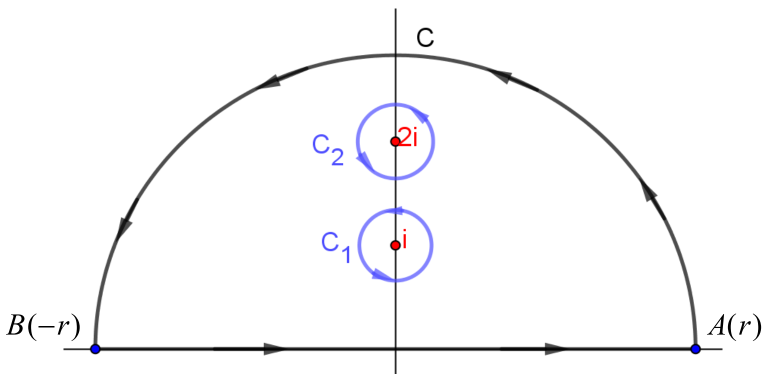
That is: $f(z) = \frac{f_1(z)}{z-i} = \frac{f_2(z)}{z-2i}$, where $f_1(z) \equiv \frac{z^2}{(z+i)(z^2+4)}$ , $f_2(z) \equiv \frac{z^2}{(z^2+1)(z+2i)}$

Since $f_1(z)$ , $f_2(z)$ are analytic, by *Cauchy*'s formula,

$$\oint_{C_1} \frac{z^2}{(z^2+1)(z^2+4)} dz = 2\pi i\, f_1(i) = 2\pi i \cdot \frac{i^2}{2i \cdot 3} = -\frac{\pi}{3},$$

$$\oint_{C_2} \frac{z^2}{(z^2+1)(z^2+4)} dz = 2\pi i\, f_2(2i) = 2\pi i \cdot \frac{(2i)^2}{(-3)\cdot 4i} = \frac{2\pi}{3}$$

$$\therefore\ I = -\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$$

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$$\boxed{I=\oint_C \frac{z^2}{(z^2+1)(z^2+4)}dz=\frac{\pi}{3} \quad \Rightarrow \quad \int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{\pi}{3}}$$

(*Continued*) This integral holds true for any $r$ $(>2)$ , thus

$$\lim_{r\to\infty}\oint_C \frac{z^2}{(z^2+1)(z^2+4)}dz=\frac{\pi}{3}$$

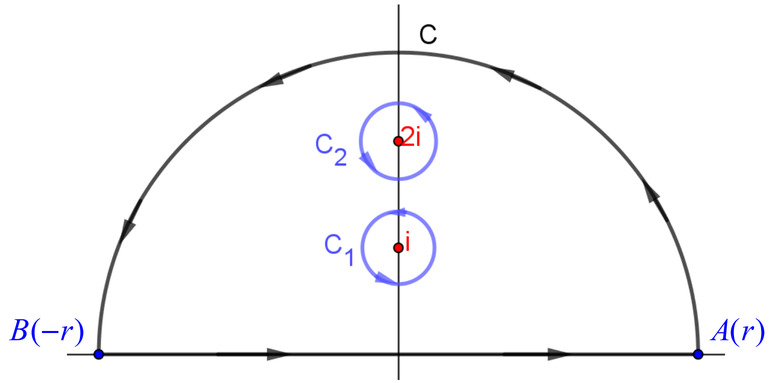
But for $z=r(\cos\theta+i\sin\theta)$ on the semicircle $\overset{\frown}{AB}$, when $r\gg 1$

$$|f(z)|\approx\frac{r^2}{r^2\cdot r^2}=\frac{1}{r^2}, \qquad |dz|=rd\theta,$$

$$\therefore \int_{\overset{\frown}{AB}}\frac{z^2}{(z^2+1)(z^2+4)}dz<\int_{\overset{\frown}{AB}}|f(z)||dz|\approx\int_0^{\pi}\frac{1}{r^2}\cdot rd\theta \xrightarrow[r\to\infty]{} 0$$

It follows that

$$\lim_{r\to\infty}\oint_C \frac{z^2}{(z^2+1)(z^2+4)}dz=\lim_{r\to\infty}\int_{\overrightarrow{BA}}\frac{z^2}{(z^2+1)(z^2+4)}dz=\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{\pi}{3}$$




What will happen when

$r\to\infty$

# Detour. *Euler*'s formula

For $\theta,\ x\ ,y \in R$

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

$$e^{x+iy} = e^{x} \cdot e^{iy} = e^{x}(\cos y + i\sin y)$$

For any $z \in C$, $e^{z}$ is defined as a power series:

$$e^{z} \equiv \sum_{n=0}^{\infty} \frac{z^{n}}{n!} = 1 + z + \frac{z^{2}}{2!} + \frac{z^{3}}{3!} + \frac{z^{4}}{4!} + \frac{z^{5}}{5!} + \frac{z^{6}}{6!} + \frac{z^{7}}{7!} + \cdots$$

Therefore for $\theta \in \mathrm{R}$,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^{2}}{2!} + \frac{(i\theta)^{3}}{3!} + \frac{(i\theta)^{4}}{4!} + \frac{(i\theta)^{5}}{5!} + \frac{(i\theta)^{6}}{6!} + \frac{(i\theta)^{7}}{7!} + \cdots \\ &= \left(1 - \frac{\theta^{2}}{2!} + \frac{\theta^{4}}{4!} - \frac{\theta^{6}}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^{3}}{3!} + \frac{\theta^{5}}{5!} - \frac{\theta^{7}}{7!} + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

[Check it with GeoGebra]

It's very easy. This will work fine.

$$e^{x} \approx \mathrm{sum}(\ \mathrm{sequence}(x^{n} / n!, n, 0, 50)\ )$$

Euler.ggb

## Example 2

$$\boxed{I = \oint_C \frac{e^{iz}}{1+z^2}dz = \frac{\pi}{e} \quad \left( \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e} \right)}$$

($C$ is the same curve as example 1, but $r > 1$)

$f(z) = \dfrac{e^{iz}}{(z-i)(z+i)}$ has a singular point $i$ inside $C$. Make a circle $C_1$ around $i$, then,

by a path deformation,

$$\oint_C \frac{e^{iz}}{1+z^2}dz = \oint_{C_1} \frac{e^{iz}}{1+z^2}dz$$

We separate $f(z)$ to analytic and divergent part.

$$f(z) = \frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{iz}/(z+i)}{z-i} = \frac{f_1(z)}{z-i}, \quad f_1(z) = \frac{e^{iz}}{z+i}$$

Because $f_1(z)$ is analytic in $C$, by *Cauchy*'s formula,

$$\oint_{C_1} \frac{e^{iz}}{1+z^2}dz = \oint_C \frac{f_1(z)}{z-i}dz = 2\pi i \times f_1(i) = 2\pi i \times \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

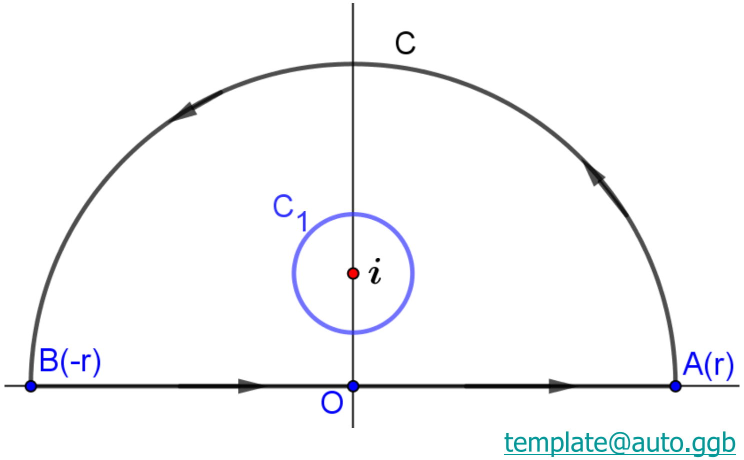
That is

$$I = \oint_C \frac{e^{iz}}{1+z^2}dz = \oint_{C_1} \frac{e^{iz}}{1+z^2}dz = \frac{\pi}{e}$$

It follows that when $r$ nears $\infty$ (same as example 1)

$$I = \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{1+x^2}dx = \frac{\pi}{e}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$$



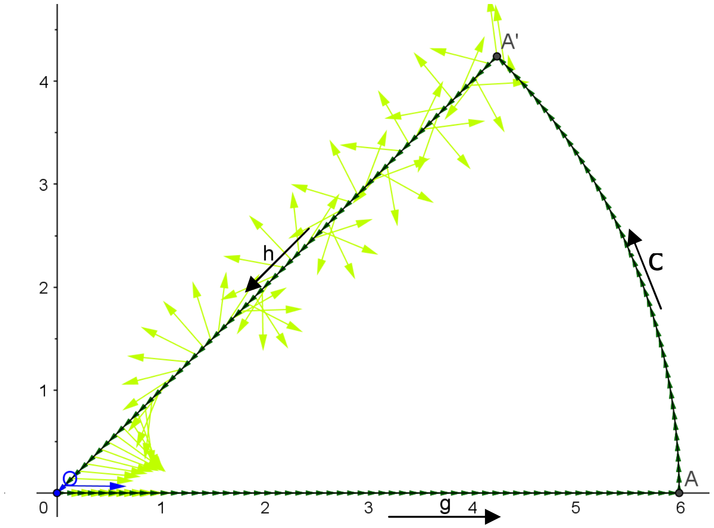
# § 9.miscellaneous

Example 3

(Fresnel's integral)

$$\oint_{g+C+h} e^{-z^2}dz = 0 \quad \Rightarrow \quad \int_0^\infty \cos(x^2)dx \;=\int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

fresnel@manual.ggb

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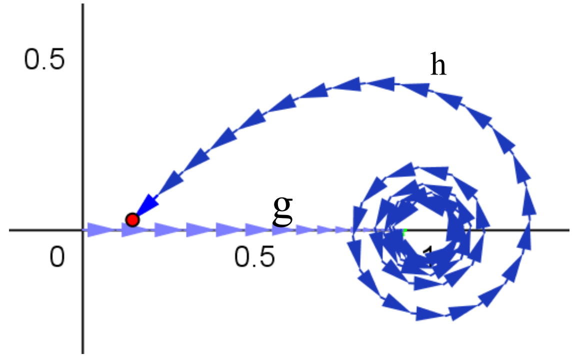
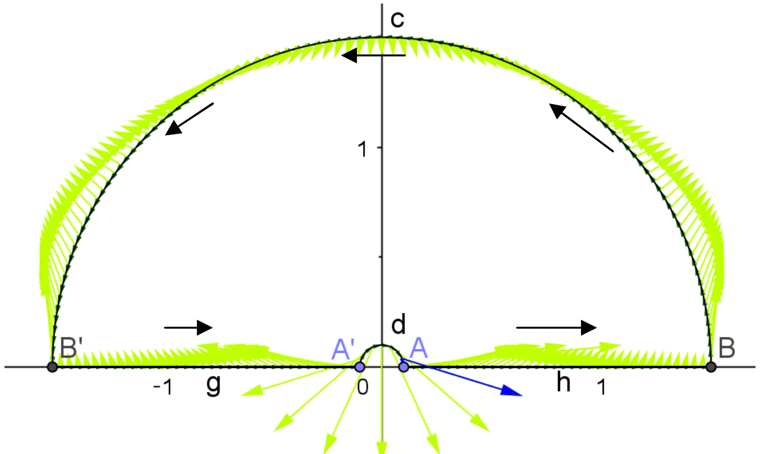



(integral on c is too tiny to see)

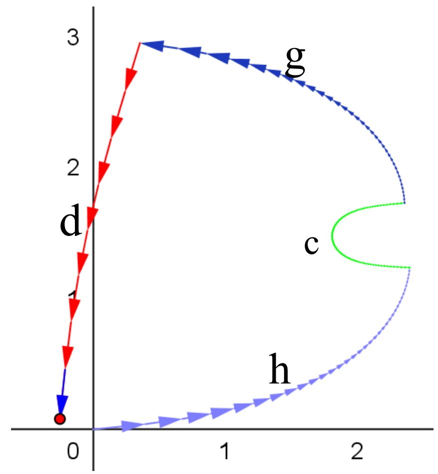
Example 4

$$\oint_{h+C+g+d} \frac{e^{iz}}{z}dz = 0 \quad \Rightarrow \quad \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$$

e^(iz)over z.ggb



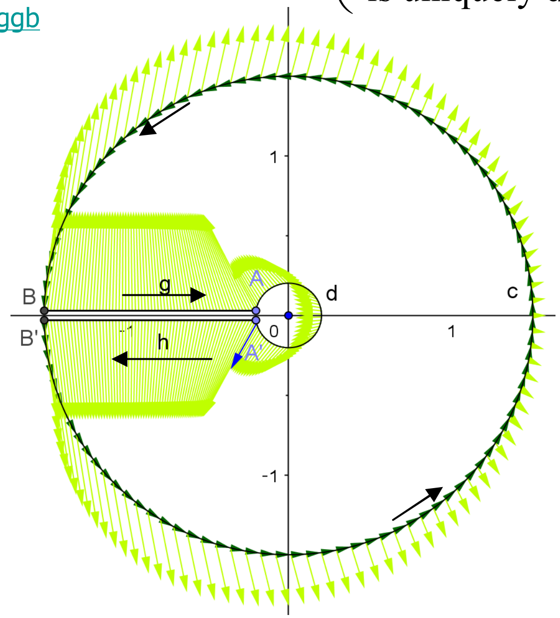

$\rightarrow$



Example 5 $\boxed{\oint_{h+C+g+d} Log\ z\ dz = 0}$

$\left( \text{When the plane is cut by the negative real axis, } Log\ z \equiv \int_1^z 1/z dz \text{ is uniquely defined and analytic in the cut plane.} \right)$

Log z@auto.ggb

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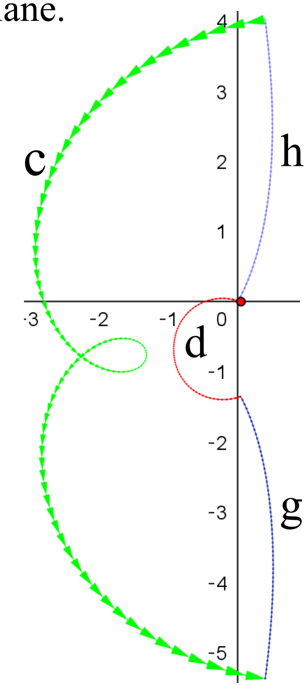


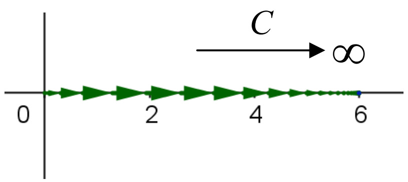

Example 6

(Γ function)

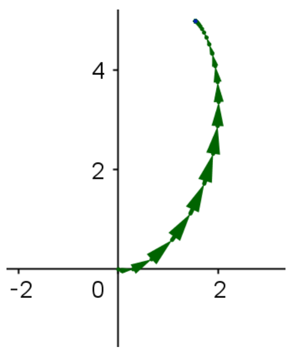
$\boxed{\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx}$

$\left( x^{s-1} \equiv e^{(s-1)\log x}, \text{ integrate complex function on the real axis} \right)$ gamma@auto.ggb

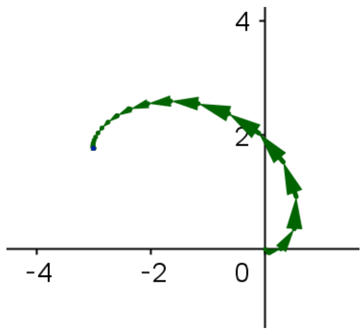
$\boxed{s = 4}$



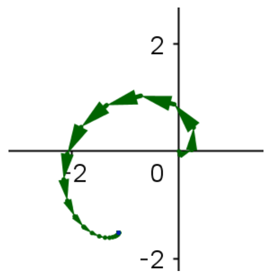

$\boxed{s = 4 + i}$



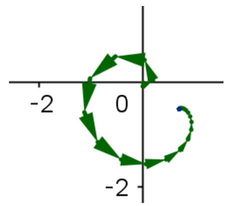
$\boxed{s = 4 + 2i}$



$\boxed{s = 4 + 3i}$



$\boxed{s = 4 + 4i}$



Example 3 $\boxed{I=\int_0^\infty \cos(x^2)dx \;=\int_0^\infty \sin(x^2)dx=\frac{\sqrt{2\pi}}{4} \quad (\textit{Fresnel}\ \text{integral})}$

Since $f(z)=e^{-z^2}$ is analytic everywhere, by Caushy's theorem,

$$\oint_{g+C+h} f(z)dz=0 \qquad \text{(g,h are segments from O to A,B, C is an arc from A to B)}$$

It is proven that ( proofs are not easy )

$$\lim_{r\to\infty}\int_C f(z)dz=0 \quad \& \quad \lim_{r\to\infty}\int_g f(z)dz=\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2} \quad \text{(Gauss integral)}$$

It follows that

$$\lim_{r\to\infty}\int_h f(z)dz=\lim_{r\to\infty}\int_g f(z)dz=\frac{\sqrt{\pi}}{2}\cdots(1)$$

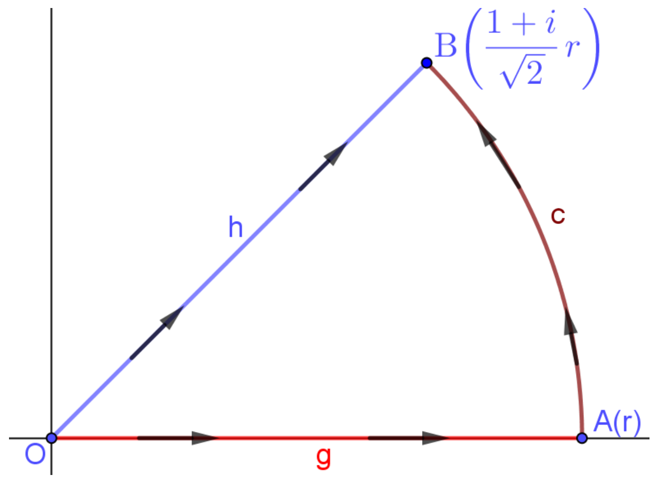
On h, $z$ is represented as $z=\frac{1+i}{\sqrt{2}}t\ (0\le t\le r)$ , then 「 $e^{-z^2}=e^{-it^2}=\cos(t^2)-i\sin(t^2),\ \ dz=\frac{1+i}{\sqrt{2}}dt$ 」

$$\therefore\ \lim_{r\to\infty}\int_h f(z)dz=\frac{1+i}{\sqrt{2}}\int_0^\infty(\cos t^2-i\sin t^2)dt=\frac{\sqrt{\pi}}{2}$$

$$\therefore\int_0^\infty(\cos t^2-i\sin t^2)dt=\frac{\sqrt{2}}{1+i}\cdot\frac{\sqrt{\pi}}{2}=\frac{\sqrt{2\pi}}{4}(1-i)$$

By comparing the real and imaginary part on both side, we get

$$\int_0^\infty \cos x^2dx \;=\int_0^\infty \sin x^2dx=\frac{\sqrt{2\pi}}{4}$$



# Rough proof of fundamental theorem

Since $F'(z) = f(z)$, when $z_{k+1} - z_k \approx 0$,

$$F(z_{k+1}) - F(z_k) \approx F'(z_k)(z_{k+1} - z_k) = f(z_k)(z_{k+1} - z_k)$$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n\text{-}1} f(z_k)(z_{k+1} - z_k)$$

$$= \sum_{k=1}^{n\text{-}1} \{F(z_{k+1}) - F(z_k)\}$$

$$= \{F(z_2) - F(z_1)\} + \{F(z_3) - F(z_2)\} + \cdots + \{F(z_n) - F(z_{n-1})\}$$

$$= F(z_n) - F(z_1)$$

# Rough proof of Cauchy's theorem

Devide the interia of a closed anticlockwised curve $C$ into many small regions $D_1, D_2, \cdots D_n$ which only share the smooth borders with neighbors, and orientalize these borders anticlockwise and name the border curve along $D_k$ *as* $C_k$.

Then on the border between $D_{k-1}$ & $D_k$, the line integral along the border is canceled. Which happens in all the borders, and thus

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

In each $D_k$, however, because $f(z)$ is analytic , when $D_k$ is very small,

$$f(z) \approx f(a_k) + (z - a_k) f'(a_k) \quad \text{(where } a_k \text{ is a fixed point in } D_k\text{)}$$

Therefore,

$$\begin{aligned}\oint_{C_k} f(z)dz &\approx \oint_{C_k} \{ f(a_k) + (z - a_k) f'(a_k) \} dz \\ &= f(a_k)\oint_C dz + f'(a_k)\oint_C (z - a_k)dz \\ &= 0\end{aligned}$$